

Cohen - Macaulay Nilpotent Schemes

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Abstract

We present here a short and partial survey about the construction and classification of the Cohen-Macaulay scheme structures on a smooth variety as support, or on a union of smooth varieties. We present, in the chronological order, the results of **Fossum** (cf. [Fo]), **Ferrand** (cf. [Fe]), **Bănică-Forster** (cf. [BF1], [BF2]) and those of [M2], [M5]. We note that in the last ten years a new terminology emerged, as for instance **ribbon** for a double structure on a scheme (usually of pure dimension 1) considered as an abstract scheme, i.e. not embedded, (there is a notion of *ribbon* also in the algebraic topology) and **rope** which is a multiple structure Y on a scheme X whose associated graded algebra has only one non-trivial component (i.e. any of the canonical filtrations is simply $Y \supset X$). The introducing of the new names created islands of research which unfortunately do not communicate among them.

KEY WORDS: Algebraic variety, algebraic scheme, Cohen-Macaulay ring, Gorenstein ring, locally complete intersection ring, dualizing sheaf, multiple structure, ribbon, rope.

1 Introduction

I was asked by the organizers of this Seminar to present a survey, for nonspecialists in algebraic geometry, of a subject which is not intuitive in the classical sense. I try here to realize this job, without giving too many formal definitions, and without complete proofs. The interested reader is invited to follow the references to fill the gaps. The examples of how to apply the general theory given in the last section are more technical, although "elementary". The multiple structures considered there are of the simplest possible kind, being *primitive* (see the definition below). In order to see more applications and examples we invite the reader to go to the papers in the *References* and to the references to those papers.

In many geometric problems, degenerated cases appear naturally, even in elementary circumstances, as, for instance, double (plane) lines as degenerated conics, sets of points among which some are "thicker" points (e.g. intersection of curves having a high contact in some points), etc. These circumstances made one of the difficulty of the classical algebraic geometry.

I will remind some questions of interest in Algebraic Geometry, where nilpotent structures arise naturally:

(i) The concept of (algebraic) schemes introduced by A. Grothendieck in the years 1950's as the next step to Serre's famous paper [Se1] made possible to attach to an algebraic (projective, affine, etc.) set different structures, so to say "according to the equations defining it". Standard nilpotent structures are the *infinitesimal neighbourhoods*. If X is an affine (or projective) scheme, defined in the affine (projective) space by an ideal I , then the scheme $X^{(1)}$ defined by I^2 is called *the first infinitesimal neighbourhood*. Also the scheme $X^{(2)}$ defined by I^3 is called the second infinitesimal neighbourhood, etc. These are given by very many equations and the multiplicity increases dramatically. We don't give here the definition of the multiplicity, but in the case of smooth X in a projective space the multiplicity of a mild (i.e. Cohen-Macaulay, see below) structure on X is the factor by which the degree of X is multiplied. This factor is a positive integer. For instance the question posed by Kronecker and Severi about the minimal number of equations defining a smooth affine space curve meant in this new frame to find a **new structure, a fortiori nilpotent** on the curve, having as few equations as possible.

(ii) The frame of schemes made possible the definition of new objects as, for instance **the Hilbert scheme**. If we consider a projective algebraic set $X \subset \mathbb{P}^n$ (zero set of some homogeneous polynomials), one can attach to it a graded ring taking the quotient of the polynomial ring by the ideal generated by all homogeneous polynomials which are zero on X . The dimension of the component of degree N , for $N \gg 0$ is polynomial in N (consequence of the Hilbert syzygies theorem). This polynomial is called *the Hilbert polynomial of X* and its properties are related to the properties of X . For instance its degree is *the dimension* of X . In the case of smooth projective curves, say over \mathbb{C} , the Hilbert polynomial has the form $\chi(N) = dN + 1 - g$, where d is the degree (it depends upon the projective embedding) and g is the *genus*, which is defined **topologically**. The set of **all** projective subschemes of \mathbb{P}^n with a fixed Hilbert polynomial is shown to be naturally an algebraic scheme, called *Hilbert scheme*. By a theorem of Hartshorne, the Hilbert schemes are connected. Intuitively, this means that one can deform a given projective subscheme of \mathbb{P}^n to any other one with the same Hilbert polynomial. But the Hilbert scheme, even in the case of curves (Hilbert polynomial of degree 1) has points which correspond to too wild objects, as for instance unions of curves and points. Even worst, the supplementary points can be *embedded*. To have an example, of what an embedded point means, consider the algebraic set X defined by the ideal $I := (x^2, xy)$ in the projective plane \mathbb{P}^2 with homogeneous coordinates x, y, z . As a set, this is the line d given by $x = 0$, but in fact the primary decomposition $I = (x) \cap (x^2, y)$ shows that "scheme-theoretically" $X = d \cup Q$, where Q is the **double point** $y = 0, x^2 = 0$.

A natural question is : is the locus of smooth curves connected ? (i.e. is the set of points in the Hilbert scheme, corresponding to the smooth curves with a fixed Hilbert polynomial connected?) An example of Gruson and Peskine (cf. [GP]) shows that for $d = 9, g = 10$ this locus has two irreducible disjoint components.

Hartshorne refined this question to the following one: If we fix a Hilbert polynomial, is the locus of the projective subschemes of \mathbb{P}^3 of (pure) dimension 1 (i.e. all the irreducible components are of dimension 1) and without **embedded points** connected ? In order to answer this question, one has to consider (mild) nilpotent curves. The answer is known to be affirmative only for $d \leq 4$, cf. [N], [NS].

(iii) A method to study vector bundles E (the method is efficient for rank two) is to consider the zero-sets of its sections (or zero-sets of a twist $E(\ell) := E \otimes \mathcal{O}(\ell)$, by a power

$\mathcal{O}(\ell) = \mathcal{O}(1)^\ell$ of the tautological bundle $\mathcal{O}(1)$. If $\ell \ll 0$, $\Gamma(E(\ell)) = 0$. If we take the smallest value of ℓ for which one has nonzero sections, the associated zero-sets will have pure codimension equal to the rank of E and will be locally complete intersection. In general one gets a scheme structure with nilpotents (cf. [H1],[M1],[BM],[HVdV],[M4],[OSz]).

(iv) One of the most challenging conjectures in the Projective Algebraic Geometry is the following one, due to Hartshorne, considered here in the special case of codimension 2:

CONJECTURE 1.1 (*Hartshorne*) *If $X \subset \mathbb{P} = \mathbb{P}^N$, $N \geq 6$ is a smooth projective variety of codimension 2, then X is (globally) complete intersection.*

Equivalent to this is the following

CONJECTURE 1.2 (*Rank 2 bundle conjecture*) *If E is a rank 2 vector bundle on $\mathbb{P} = \mathbb{P}^N$, $N \geq 6$, then E splits.*

Surprisingly enough, equivalent to these two is the following one:

CONJECTURE 1.3 *If Y is a multiplicity 3 structure (see the definition bellow) on a linear subspace $\mathbb{P} = \mathbb{P}^N$ of $\mathbb{P}^{N+\delta}$, contained in the first infinitesimal neighbourhood of \mathbb{P} , then there exists a double structure Z on \mathbb{P} contained in Y .*

For other equivalent variants see [V2].

2 Fossum - Ferrand doubling

2.1 The algebraic case (Fossum)

For us all the rings R will be local commutative noetherian algebras over an algebraically closed field k . In this case a local ring R is called Cohen-Macaulay iff the maximal ideal m_R of R contains $n = \text{dimension}(R)$ elements x_1, \dots, x_n which "behave as *indeterminates*". Here a *sequence of elements* x_1, \dots, x_m *behaves as indeterminates* means

(a) *the homomorphism sending X_i in x_i is a faithfully flat injection $k[X_1, \dots, X_m] \rightarrow R$ from the ring of polynomials in m indeterminates into R*

or, equivalently:

(b) *x_1 is not a zero-divisor in R , x_2 is not a zero-divisor in R/x_1R , \dots , x_m is not a zero-divisor in $R/(x_1R + \dots + x_{m-1}R)$.*

A sequence of such elements $x_1, \dots, x_m \in m_R$, $m \leq n$, is called *regular*.

Over a ring R , the functor of taking the dual with respect to R (i.e. $\text{Hom}(\cdot, R)$) does not have the properties which we would require for a good duality. The Cohen-Macaulay rings have a *dualizing module* (called also *canonical module*) E . A canonical module E over a commutative noetherian local ring R is a finitely generated module with the properties:

- (a) the canonical morphism $R \rightarrow \text{Hom}_R(E, E)$ is bijective
- (b) for all $i > 0$, $\text{Ext}_R^i(E, E) = 0$
- (c) E has finite injective dimension.

One shows also that, conversely, if R has a canonical module, then R is Cohen-Macaulay. Of special interest are the rings which are canonical modules over themselves. They are called *Gorenstein rings*. Regular rings and complete intersection rings are Gorenstein.

One shows that a quotient S of a Gorenstein ring R is Cohen-Macaulay iff $\text{Ext}_R^i(S, R) = 0$ for all $i \neq d := \dim R - \dim S$. In this case the canonical module of S is $\text{Ext}_R^d(S, R)$.

Reiten proved in [R] the following:

THEOREM 2.1 *Let S be a Cohen-Macaulay ring and M a canonical S -module. Then the trivial extension $M \rtimes S$ of S by M is Gorenstein. The trivial extension means the ring structure on $M \times S$ given by $(s_1, m_1)(s_2, m_2) = (s_1 m_2 + s_2 m_1, s_1 s_2)$.*

□

More generally, if S is a ring and M a S -module, a *commutative extension of S by M* is an exact sequence:

$$0 \rightarrow M \xrightarrow{i} B \xrightarrow{p} S \rightarrow 0 ,$$

B being a commutative ring, p a homomorphism of rings, and for all $b \in B$ and $m \in M$: $bi(m) = i(p(b)m)$. Via i , M is identified with an ideal in B , whose square is zero. In fact, if B is a ring and I is a square zero ideal of B , then we can write the above extension as

$$0 \rightarrow I \xrightarrow{i} B \xrightarrow{p} B/I \rightarrow 0 ,$$

I being an R/I -module. Abusively, we call also the ring B a *commutative extension of S by M* .

Fossum proved in [Fo] the following:

THEOREM 2.2 *A commutative extension:*

$$0 \rightarrow E \rightarrow B \rightarrow S \rightarrow 0$$

of a Cohen-Macaulay ring S by a canonical S -module E is a Gorenstein ring.

□

Remark 2.3 *The multiplicity of the extension is double the multiplicity of the ring S , since the multiplicity of the canonical module equals the multiplicity of the ring. This motivate us to call an extension like above a **Fossum doubling**.*

We can construct embedded doublings. If we take $B := R/J$ and $S := R/I$ as quotients of a (say, regular) ring R , then an extension as above can be written:

$$0 \rightarrow I/J \xrightarrow{i} R/J \xrightarrow{p} R/I \rightarrow 0 ,$$

where $I^2 \subset J$.

The above explanations show that, given $S = R/I$ a Cohen-Macaulay ring, quotient of the ring R , and $\pi : I/I^2 \rightarrow E_S$ a surjection to a canonical S -module, a commutative extension of S by E_S can be produced taking the kernel of π :

$$0 \rightarrow J/I^2 \rightarrow I/I^2 \xrightarrow{\pi} E_S \rightarrow 0 .$$

The doubling of Fossum can be applied to schemes, *abstractly*, to obtain the so-called **ribbons** or *embedded*, to obtain the so-called Ferrand's doubling. It is beyond the aim of this paper to speak about ribbons. To make the reader curious, we quote from [BE]: "The theory of ribbons is in some respects remarkably close to that of smooth curves, but ribbons are much easier to construct and work with."

2.2 The geometric doubling (Ferrand)

Consider now X a locally Cohen-Macaulay subscheme of a regular one Z . Then there is a dualizing \mathcal{O}_X -module ω_X which gives a good duality on the category of coherent sheaves on X . Locally, ω_X corresponds to the canonical module of the respective local rings of X . The method of Fossum can be then globalized as follows:

CONSTRUCTION 2.4 *Let X be a Cohen-Macaulay subscheme of the regular scheme Z . Let I be the ideal (sheaf ideal, of course) defining X in Z , L a line bundle (locally free sheaf of rank 1) on X and $p : I/I^2 \rightarrow \omega_X \otimes L$ a surjection. Then the kernel of p is the of the form J/I^2 , where J is the ideal of a Gorenstein scheme structure Y in which X is a closed subscheme. As sets, $Y = X$.*

This is the *Ferrand's doubling method*, [Fe]. We can write the exact sequence defining J also:

$$0 \rightarrow L \otimes \omega_X \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0 .$$

This exact sequence is the result of *patching together* exact sequences of the Fossum's type.

In fact, Ferrand considers the case:

THEOREM 2.5 *In the above construction $Z = \mathbb{P}^3$, L induced, i.e. $L = \mathcal{O}_X(n) := \mathcal{O}_{\mathbb{P}^3}(n)|_X$. If $p : I \rightarrow \omega_X(n)$ is a surjection such that $H^1(\mathbb{P}^3, I(-n)) \rightarrow H^1(X, \omega_X)$ vanishes, then the curve Y defined by $J = \ker(p)$ is the scheme of zeros of a rank 2 vector bundle, $\omega_Y \cong \mathcal{O}_Y(-n)$ and $\text{Ann}(I/J) = I$.*

It is interesting to quote Ferrand (cf. [Fe]): "L'idée d'imposer la condition $\text{Ann}(I/J) = I$ provient de la théorie de la liaison de Peskine et Szpiro, [PS]". (For further connections with the linkage theory see the next sections.)

He combines the above construction with the following known facts:

1) *a locally Cohen-Macaulay curve Y in \mathbb{P}^3 is the scheme of zeros of a rank 2 vector bundle E iff $\omega_Y \cong \mathcal{O}_Y(m)$ for some $m \in \mathbb{Z}$. Y is globally complete intersection iff E splits (cf. [Se2].*

2) *If E is a vector bundle of rank at least 2 on a curve X , generated by its global sections, then there is a section $\mathcal{O}_X \rightarrow E$ which vanishes nowhere.*

to obtain

THEOREM 2.6 *If X is a locally complete intersection curve in \mathbb{P}^3 , defined by the ideal I , then there exists a curve Y , containing X , which is the scheme of zeros of a section of a rank 2 vector bundle E on \mathbb{P}^3 . The ideal of Y satisfies $\text{Ann}(I/J) = I$.*

If we apply the above result for the affine space $Z = \mathbb{A}^3$, (cf. [Sz]), as any vector bundle on \mathbb{A}^3 is trivial (cf. [Su], [Q]) one answers affirmatively a question of Kronecker and Severi:

THEOREM 2.7 (*Ferrand-Szpiro*)(cf. [Sz]) *Any locally complete intersection curve in \mathbb{A}^3 is set-theoretically (i.e. changing conveniently the scheme structure) a complete intersection.*

Remark 2.8 *The multiplicity of the local ring of Y in a point is double of that of the same point considered on X . In particular, when X is smooth the scheme Y has multiplicity 2 in any point.*

3 Higher multiplicities

The first description of the locally complete intersection scheme structures on a line in \mathbb{P}^3 with multiplicity 3 was given by Hartshorne, [H1].

The first systematic study of multiple structures after the papers of Fossum, Ferrand was done by Bănică și Forster in [BF1]. Unfortunately this paper had a small circulation. Here the classification up to multiplicity 4 is done, for scheme structures Y on a smooth curve X as suport, embedded in a smooth threefold Z . The idea was to cut (locally) the multiple structure with a transverse smooth germ of a surface. One obtains a multiple point in $\mathbb{C}\{x, y\}$. These are classified (cf. [Br]). Deforming such a multiple point, one gets a *germ of a multiple curve*. The local results were then patched together to give complete constructions of the global multiple structures. The main results of [BF1] were included in [BF2], where a new idea was introduced. Namely, here the Cohen-Macaulay stratification, which was present also in [BF1], was constructed directly: if we cut the **thick** scheme structure Y on X with the successive infinitesimal neighbourhoods $X^{(i)}$ of X (defined by the powers I^i of the ideal I of X in Z) and **throw away the embedded points**, (in dimension 1, Cohen-Macaulay property is equivalent with the lack of embedded points), one gets a canonical filtration of Y with Cohen-Macaulay curves. The possible filtrations are classified and one constructs conversely, step by step, from the reduced structure X the thicker ones.

Independently (of [BF2]), in [M2] another Cohen-Macaulay stratification was proposed, inspired from the linkage theory of Peskine and Szpiro [PS]. Namely, if J is the ideal of the thick scheme Y with X as support, one considers the schemes defined by $J : I^i$. Observe that this is not a genuine linkage, because in general $J \not\subset I^i$. This was shown to work for curves and even more, for any dimension of X and any dimension of Z , in the case of locally complete intersection multiple structure on smooth support, up to multiplicity 4. The locally complete intersection (lci, for short) scheme structures **in any dimension and any codimension, any characteristic of the base field** were classified, up to multiplicity 4.

The two filtrations, from [BF2] and [M2] differ in general. For both of them the successive quotients of the defining ideals are, in good circumstances (e.g. *on smooth support*) **vector bundles** on X . So, in principle if one takes successively surjections $p_1 : I \rightarrow E_1$, $p_2 : \ker(p_1) \rightarrow E_2, \dots, p_m : \ker(p_{m-1}) \rightarrow E_m$, where E_j are arbitrary vector bundles, one gets all the thick structures in the asserted range, and a Cohen-Macaulay filtration.

The problem is that such a procedure is by no means canonical and the counting of parameters is difficult. On the contrary, the filtrations considered in [BF2] and [M2] are canonical. The one coming from the linkage theory has the advantage that provides directly some exact sequences. Also it can be applied to nonirreducible schemes. On the other hand the Bănică-Forster filtration has the advantage of a multiplicative structure. Namely, if $J_i :=$ the ideal of the scheme obtained throwing away the embedded points of $Y \cap X^{(i)}$, then the graded object (with finitely many graded components) $\bigoplus J_i/J_{i+1}$ is a graded \mathcal{O}_X -algebra, each piece of which is a vector bundle.

The next step was done in [M5], where one shows that the filtration from [M2] works in fact up to multiplicity 6 (**any dimension, any codimension**) for lci structures. In [M5] a new filtration was introduced, namely $J : (J : I^i)$. The associated graded object is a graded \mathcal{O}_X -algebra, $\mathcal{A}(Y)$. The graded object $\mathcal{M}(Y)$ associated to the filtration $J : I^i$ has a natural structure of graded $\mathcal{A}(Y)$ -module. We denote by $\mathcal{B}(Y)$ the graded algebra coming from the Bănică-Forster filtration. In general $\mathcal{B}(Y)$, $\mathcal{A}(Y)$, $\mathcal{M}(Y)$ are different. The multiplicative structure is always nontrivial (of course, if the graded components involved are not zero). There are canonical homomorphisms $\mathcal{B}(Y) \rightarrow \mathcal{A}(Y) \rightarrow \mathcal{M}(Y)$. In the rest of this section we discuss only the simplest constructions from [M5]. When one of the above graded objects has line bundles as components, then all three filtrations are equal. Such a structure was called *quasiprimitive* in [BF1], [BF2]. One shows in this case that, in general: $\mathcal{B}(Y) \cong \mathcal{A}(Y) \cong \mathcal{M}(Y) \cong \bigoplus_{i=1}^{i=\mu-1} L^i(D_i)$, where D_i are effective divisors on X and L is a line bundle on X . Bănică-Forster called a multiple structure with all $D_i = 0$ a *primitive one*. The fact that the multiplicative structure is not trivial implies inequalities fulfilled by D_i . The requirement that the structure is locally complete intersection restricts even more the shape of $\mathcal{A}(Y)$. The primitive structures are always lci (locally complete intersection). For the multiplicity $\mu = 3$ no other quasiprimitive structure is lci. For $\mu = 4$ there is only one possible shape of $\mathcal{A}(Y)$, besides the primitives one: $\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2(D) \oplus L^3(D)$. For $\mu = 5$ there is also only one possibility: $\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2(D) \oplus L^3(2D) \oplus L^4(2D)$. For multiplicity 6 the situation is more complicated: there are three models:

$$\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2 \oplus L^3(D) \oplus L^4(D) \oplus L^5(D) ,$$

$$\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2(E) \oplus L^3(E) \oplus L^4(2E) \oplus L^5(2E) ,$$

$$\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2(F_1) \oplus L^3(F_1 + F_2) \oplus L^4(2F_1 + F_2) \oplus L^5(2F_1 + F_2) .$$

which can be *combined*, in the following sense: Consider D, E, F_1, F_2 effective divisors on X , such that D, E and $F_1 + F_2$ are pairwise disjoint. Then outside the support of E and $F_1 + F_2$ the shape is of the *type* D , outside D and $F_1 + F_2$ is of *type* E , etc. An example of such a combination is:

$$\mathcal{A}(Y) = \mathcal{O}_X \oplus L \oplus L^2(E) \oplus L^3(D + E) \oplus L^4(D + 2E) \oplus L^5(D + 2E)$$

It is beyond the aim of this survey to present in detail the results of [BF1], [BF2] or [M2], [M5].

4 Old and New Examples

4.1 Smooth Hilbert scheme in a point corresponding to double structures

In [BM] we proved the smoothness of the moduli space of stable rank 2 vector bundles on \mathbb{P}^3 with Chern classes $c_1 = -1$, $c_2 = 4$, in the points corresponding to those vector bundles which correspond to double structures on conics. A step in this proof was the fact that the Hilbert scheme \mathcal{H}_{4t+6} of closed subschemes of \mathbb{P}^3 with Hilbert polynomial $4t + 6$ is smooth in the points corresponding to double conics. **The same proof**, which we repeat here *mutatis mutandi*, in the longer variant from [BM'], works for the following general case:

THEOREM 4.1 *The Hilbert scheme \mathcal{H}_{4t+r+2} , of closed subschemes in \mathbb{P}^3 with Hilbert polynomial $4t + r + 2$, $r > 0$, is smooth in the points corresponding to double conics and the set \mathcal{H}_d corresponding to double conics is open in \mathcal{H}_{4t+r+2} . The closure of \mathcal{H}_d is an irreducible component of the Hilbert scheme.*

Proof. Let C be a smooth conic in \mathbb{P}^3 . Let Y be a doubling given by an extension:

$$0 \rightarrow L \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_C \rightarrow 0 ,$$

or, equivalently, by an exact sequence:

$$0 \rightarrow I_Y/I_C^2 \rightarrow I_C/I_C^2 \rightarrow L \rightarrow 0 ,$$

where L is a line bundle on C , which restricted to \mathbb{P}^1 via $\mathbb{P}^1 \xrightarrow{i} C \hookrightarrow \mathbb{P}^3$ is $\mathcal{O}_{\mathbb{P}^1}(r)$. As $i^*\mathcal{O}_C(\ell) = \mathcal{O}_{\mathbb{P}^1}(2\ell)$, the Hilbert polynomial of Y is $\chi(\mathcal{O}_Y(t)) = \chi(\mathcal{O}_C(t)) + \chi(L) = (2t + 1) + (2t + r + 1) = 4t + r + 2$. As $H^1(C, L) = H^1(\mathbb{P}^1, \mathcal{O}(r)) = 0$, one has $\text{Pic}(Y) \cong \text{Pic}(C)$. Note that $\omega_Y|_C \cong \omega_C \otimes L^{-1}$ and then $\omega_Y|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-r - 2)$.

To double the conic C "with L " is equivalent to giving $a \in H^0(L(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(r + 2))$ and $b \in H^0(L(2)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(r + 4))$ without common zeros on C , the pair (a, b) being unique for each Y with support C , up to a factor in \mathbb{C}^* . (When r is even, $r = 2R$, is very easy to write the equations of Y . The ideal of Y is of the shape $(Ah + B\tau, h^2, h\tau, \tau^2)$, where $A \in H^0(L(1)) = H^0(\mathcal{O}_C(R + 1))$, $B \in H^0(L(2)) = H^0(\mathcal{O}_C(R + 2))$ have no common zeros along C and where (h, τ) are the equations which give C in \mathbb{P}^3 .) Consider the family $\mathcal{H}^* = \{H, C, (a, b)\}/\mathbb{C}^*$, where H is a plane in \mathbb{P}^3 , C a conic in it and (a, b) are the above elements, identified with homogeneous polynomial in two variables, of degrees $r + 4$, respectively $r + 3$, without common factors, modulo a factor in \mathbb{C}^* . \mathcal{H}^* has a natural structure of an algebraic variety, which is connected, smooth, quasiprojective, rational, of dimension $2r + 15$ (8 parameters for the conic and $2r + 7$ for the doubling). The map $\mathcal{H}^* \rightarrow \mathcal{H}_{4t+r+2}$ which associate to $(H, C, (a, b))$ the doubling of C with data (a, b) is clearly algebraic. It is easy to see that it is also bijective onto the locus $\mathcal{H}_d \subset \mathcal{H}_{4t+r+2}$ of double conics (cf. [BM'], Lemma 5). To show that $\mathcal{H}^* \cong \mathcal{H}_d$ we have to show that the tangent space at any point y corresponding to double structure Y on a conic C has dimension $2r + 15$. But, if \mathcal{H} denotes our Hilbert scheme, $\dim T_{\mathcal{H}, y} = h^0(N_Y|_{\mathbb{P}^3}) = h^0((I_Y/I_Y^2)^\vee) = h^1((I_Y/I_Y^2) \otimes \omega_Y)$. We shall use the exact sequences:

$$0 \rightarrow (I_C I_Y / I_Y^2) \otimes \omega_Y \rightarrow (I_Y / I_Y^2) \otimes \omega_Y \rightarrow (I_Y / I_C I_Y) \otimes \omega_Y \rightarrow 0 \quad (1)$$

$$0 \rightarrow ((I_Y^2 + I_C^3)/I_Y^2) \otimes \omega_Y \rightarrow (I_C I_Y/I_Y^2) \otimes \omega_Y \rightarrow (I_C I_Y/(I_Y^2 + I_C^3)) \otimes \omega_Y \rightarrow 0 \quad (2)$$

$$0 \rightarrow (I_C^2/I_C I_Y) \otimes \omega_Y \rightarrow (I_Y/I_C I_Y) \otimes \omega_Y \rightarrow (I_Y/I_C^2) \otimes \omega_Y \rightarrow 0 \quad (3)$$

From the exact sequence:

$$0 \rightarrow I_Y/I_C^2 \rightarrow I_C/I_C^2 (= \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2)) \rightarrow I_C/I_Y (= L) \rightarrow 0$$

one deduces

$$I_Y/I_C^2 \cong L^{-1}(-3) .$$

This together with $I_C^2/I_C I_Y \cong (I_C/I_Y)^{\otimes 2} = L^2$ and $\omega_Y|_C \cong L^{-1}(-1)$ allow us to rewrite (3):

$$0 \rightarrow L^1(-1) \rightarrow (I_Y/I_C I_Y) \otimes \omega_Y \rightarrow L^{-2}(-4) \rightarrow 0 \quad (4)$$

In (2) we need to compute the middle term. It is easy to see that the left term is given by $I_Y^2 + I_C^3/I_Y^2 \cong I_C^3/I_C^2 I_Y \cong (I_C/I_Y)^{\otimes 3} = L^3$, and so:

$$I_Y^2 + I_C^3/I_Y^2 \otimes \omega_Y \cong L^3 \otimes \omega_Y \cong L^3 \otimes \omega_Y|_C \cong L^2(-1) .$$

For the right term we use the exact sequence:

$$0 \rightarrow ((I_Y^2 + I_C^3)/I_C^3) \rightarrow (I_C I_Y/I_C^3) \rightarrow (I_C I_Y/(I_Y^2 + I_C^3)) \rightarrow 0 ,$$

where the first term is: $(I_Y^2 + I_C^3)/I_C^3 \cong I_Y^2/I_Y^2 \cap I_C^3 \cong I_Y^2/I_Y I_C^2 \cong (I_Y/I_C^2)^{\otimes 2} \cong L^{-2}(-6)$, and were we need also $\det(I_C I_Y/I_C^3)$. The exact sequence:

$$0 \rightarrow I_C I_Y/I_C^3 \rightarrow I_C^2/I_C^3 (\cong \mathcal{O}_C(-2) \oplus \mathcal{O}_C(-3) \oplus \mathcal{O}_C(-4)) \rightarrow I_C^2/I_C I_Y (\cong L^2) \rightarrow 0$$

implies $\det(I_C I_Y/I_C^3) \cong L^{-2}(-9)$, so that $I_C I_Y/(I_Y^2 + I_C^3) \cong \mathcal{O}_C(-3)$, and then

$$(I_C I_Y/(I_Y^2 + I_C^3)) \otimes \omega_Y \cong \mathcal{O}_C(-3) \otimes \omega_Y \cong L^{-1}(-4)$$

The exact sequence (2) becomes:

$$0 \rightarrow L^2(-1) \rightarrow (I_C I_Y/I_Y^2) \otimes \omega_Y \rightarrow L^{-1}(-4) \rightarrow 0 \quad (5)$$

Applying the long exact sequences of cohomology to (4), noticing:

$$H^0(L^{-2}(-4)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2r-8)) = 0 , \quad (6)$$

$$H^1(L^{-2}(-4)) \cong H^1(\mathbb{P}^1, \mathcal{O}(-2r-8)) \cong \mathbb{C}^{2r+7} , \quad (7)$$

$$H^0(L(-1)) \cong H^0(\mathbb{P}^1, \mathcal{O}(r-2)) = \mathbb{C}^{r-1} , \quad (8)$$

$$H^1(L(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}(r-2)) \cong 0 , \quad (9)$$

one gets:

$$H^0((I_Y/I_C I_Y) \otimes \omega_Y) \cong \mathbb{C}^{r-1}, \quad (10)$$

$$H^1((I_Y/I_C I_Y) \otimes \omega_Y) \cong \mathbb{C}^{2r+7}. \quad (11)$$

Now apply the long exact sequence of cohomology to (5), noticing:

$$H^0(L^2(-1)) \cong H^0(\mathbb{P}^1, \mathcal{O}(2r-2)) = \mathbb{C}^{2r-1}, \quad (12)$$

$$H^1(L^2(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}(2r-2)) = 0, \quad (13)$$

$$H^0(L^{-1}(-4)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-r-8)) = 0, \quad (14)$$

$$H^1(L^{-1}(-4)) \cong H^1(\mathbb{P}^1, \mathcal{O}(-r-8)) \cong \mathbb{C}^{r+7}, \quad (15)$$

and get:

$$H^0((I_C I_Y/I_Y^2) \otimes \omega_Y) \cong \mathbb{C}^{2r-1} \quad (16)$$

$$H^1((I_C I_Y/I_Y^2) \otimes \omega_Y) \cong \mathbb{C}^{r+7} \quad (17)$$

The long exact sequence of cohomology applied to (1) gives:

$$\begin{aligned} 0 \rightarrow H^0((I_C I_Y/I_Y^2) \otimes \omega_Y) &\rightarrow H^0((I_Y/I_Y^2) \otimes \omega_Y) \rightarrow H^0((I_Y/I_C I_Y) \otimes \omega_Y) \rightarrow \\ &H^1((I_C I_Y/I_Y^2) \otimes \omega_Y) \rightarrow H^1((I_Y/I_Y^2) \otimes \omega_Y) \rightarrow H^1((I_Y/I_C I_Y) \otimes \omega_Y) \rightarrow 0, \end{aligned}$$

and the above computations imply that $h^1((I_Y/I_Y^2) \otimes \omega_Y) = 2r+15$ is equivalent to the injectivity of the map

$$H^0((I_Y/I_C I_Y) \otimes \omega_Y) \rightarrow H^1((I_C I_Y/I_Y^2) \otimes \omega_Y).$$

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (I_C I_Y/I_Y^2) \otimes \omega & \longrightarrow & (I_Y/I_Y^2) \otimes \omega & \longrightarrow & (I_Y/I_C I_Y) \otimes \omega \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & (I_C I_Y/I_Y^2 + I_C^3) \otimes \omega & \longrightarrow & (I_Y/I_Y^2 + I_C^3) \otimes \omega & \longrightarrow & (I_Y/I_C I_Y) \otimes \omega \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (I_C I_Y/I_Y^2 + I_C^3) \otimes \omega & \longrightarrow & (I_C^2/I_Y^2 + I_C^3) \otimes \omega & \longrightarrow & (I_C^2/I_C I_Y) \otimes \omega \longrightarrow 0 \end{array}$$

which gives the commutative diagram of cohomology:

$$\begin{array}{ccc} H^0((I_Y/I_C I_Y) \otimes \omega) & \longrightarrow & H^1((I_C I_Y/I_Y^2) \otimes \omega) \\ \parallel & & \downarrow \wr \\ H^0((I_Y/I_C I_Y) \otimes \omega) & \longrightarrow & H^1((I_C I_Y/I_Y^2 + I_C^3) \otimes \omega) \\ \uparrow \wr & & \parallel \\ H^0((I_C^2/I_C I_Y) \otimes \omega) & \longrightarrow & H^1((I_C I_Y/I_Y^2 + I_C^3) \otimes \omega) \end{array}$$

such that we have to show that the lower row is injective, or equivalently $H^0((I_C^2/I_Y^2 + I_C^3) \otimes \omega_Y) = 0$.

The locally free \mathcal{O}_C -module $I_C^2/I_Y^2 + I_C^3$ can be computed from the exact sequence of \mathcal{O}_C -modules:

$$0 \rightarrow (L^{-2}(-6) \cong (I_Y^2 + I_C^3)/I_C^3 \rightarrow I_C^2/I_C^3 \cong \mathcal{O}_C(-4) \oplus \mathcal{O}_C(-3) \oplus \mathcal{O}_C(-2)) \rightarrow I_C^2/(I_Y^2 + I_C^3) \rightarrow 0 ,$$

which pulled-back on \mathbb{P}^1 is isomorphic to:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2r-12) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(-8) \oplus \mathcal{O}_{\mathbb{P}^1}(-6) \oplus \mathcal{O}_{\mathbb{P}^1}(-4) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(r-4) \oplus \mathcal{O}_{\mathbb{P}^1}(r-2) \rightarrow 0 ,$$

where $\alpha = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 2b & -a & 0 \\ 0 & -b & 2a \end{pmatrix}$. Then $i^*(I_C^2/(I_Y^2 + I_C^3) \otimes \omega_Y) \cong \mathcal{O}_{\mathbb{P}^1}(-6) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$, and so $H^0((I_C^2/I_Y^2 + I_C^3) \otimes \omega_Y) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-6) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)) = 0$. \square

*Remark 4.2 (i) The fact that \mathcal{H}_d is an irreducible component of the Hilbert scheme was reobtained in [NS], where, more than this, **the whole decomposition in irreducible components of the Hilbert scheme** is described and, more difficult, **the connectedness of the Hilbert scheme of curves of degree 4** is proved. The dimension of \mathcal{H}_d follows also from [HS].*

(ii) It is not true in general that double structures on plane curves in \mathbb{P}^3 give smooth points of the respective Hilbert scheme. (cf. [BM']).

(iii) The case of ropes on lines is analyzed in [NNS] and a smoothness result is obtained (cf. loc. cit. for the exact statement). Although in [NNS] the method is not the computing of the global sections of the normal sheaf, some parallel to our old proof can be observed (e.g the use of matrices similar to our α, β).

4.2 A Double or a Triple Plane in \mathbb{P}^5 is Never the Scheme of Zeros of a Section of an Indecomposable Vector Bundle on \mathbb{P}^5

We mentioned that any smooth curve in \mathbb{P}^3 is the support of a locally complete structure of double degree which is the scheme of zeroes of a section of a convenient rank 2 indecomposable vector bundle on \mathbb{P}^3 . For surfaces in \mathbb{P}^5 this is no more true. We show:

THEOREM 4.3 *Let $\mathbb{P}^2 \cong X \subset \mathbb{P}^5$ a plane in \mathbb{P}^5 . There is no indecomposable rank three vector bundle F on \mathbb{P}^5 with a section having as vanishing scheme a double structure on X .*

Proof. Suppose such a doubling Y exists. After [M2] a double structure on a plane is given by a (Fossum-)Ferrand construction. Then we should have an exact sequence:

$$0 \rightarrow \mathcal{O}_X(r) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0 ,$$

where $r \geq 0$ (for $r = -1$ the doubling is globally complete intersection, so that F exists and splits). If Y is the scheme of zeros of a section $s \in H^0(F)$ in a rank 3 vector bundle

F , and $E := F^\vee$, then one has an exact sequence:

$$0 \rightarrow \Lambda^3 E \rightarrow \Lambda^2 E \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^5} \rightarrow \mathcal{O}_Y \rightarrow 0 .$$

The doubling exists for each r in our range because there are surjections

$$I_X/I_X^2 \cong 3\mathcal{O}_X(-1) \xrightarrow{p} \mathcal{O}_X(r) ,$$

such a surjection being determined by three forms of degree $r+1$ in the homogeneous coordinates on $X = \mathbb{P}^2$, without common zeros. (We used tacitely the notation of the type $3A$ for $A \oplus A \oplus A$.)

The plan of the proof is the following: we compute the Hilbert polynomial of Y from the two above exact sequences, as a function in r and as a function in the Chern classes of E (one uses Riemann-Roch-Hirzebruch Theorem, cf. [Hi]). This will determine the Chern classes of E with respect to r and then we see that E does not satisfy the Schwarzenberger conditions (cf. *Appendix One* by R. L. E. Schwarzenberger to [Hi] — they come from the requirement that the Hilbert-Euler characteristic of E takes values in \mathbb{Z}).

Denote by c_1, c_2, c_3 the Chern classes of E . Then $\Lambda^3 E \cong \mathcal{O}_{\mathbb{P}^5}(c_1)$, $\Lambda^2 E$ has Chern classes : $c'_1 = 3c_1$, $c'_2 = c_1^2 + c_2$, $c'_3 = c_1 c_2 - c_3$. After a long computation, which we don't reproduce here, one gets the Hilbert polynomial of Y :

$$\chi_Y(t) = -\frac{c_3}{2}t^2 - \frac{(c_1 + 6)c_3}{2}t + \frac{(c_2 - 2c_1^2 - 18c_1 - 51)c_3}{2}$$

On the other hand the defining extension of \mathcal{O}_Y gives:

$$\chi_Y(t) = \chi_{\mathbb{P}^2}(t) + \chi_{\mathbb{P}^2}(t+r) = \binom{t+2}{2} + \binom{t+r+2}{2} = t^2 + (r+3)t + \frac{r^2 + 3r + 4}{2}$$

Comparing the two formulas one gets:

$$\begin{aligned} c_1 &= r - 3 , \\ c_2 &= \frac{3r^2 + 9r + 26}{2} , \\ c_3 &= -2 . \end{aligned}$$

We show now that there is no value of r for which the above Chern classes can be attained. Namely, we write the Hilbert polynomial of E in the basis $\binom{t+i}{i}$, $i = 1, \dots, 5$:

$$\begin{aligned} \chi_E(t) &= 3\binom{t+5}{5} - (r-3)\binom{t+4}{4} + (r^2 + 7r + 10)\binom{t+3}{3} + \\ &\quad \frac{7r^3 + 30r^2 + 29r - 54}{12}\binom{t+2}{2} + \frac{r^4 - 24r^3 - 197r^2 - 560r - 548}{48}\binom{t+1}{1} \\ &\quad - \frac{19r^5 + 235r^4 + 1305r^3 + 3765r^2 + 5616r + 3140}{480} \end{aligned} \quad (18)$$

The polynomial $\chi_E(t)$ takes integer values for $t \in \mathbb{Z}$ iff the coefficients are in \mathbb{Z} . In particular we should have:

$$\begin{aligned} 7r^3 + 30r^2 + 29r - 54 &\equiv 0 \pmod{3} \\ r^4 - 24r^3 - 197r^2 - 560r - 548 &\equiv 0 \pmod{3} \end{aligned}$$

what is impossible.

Remark 4.4 The above principally easy observation should be very well known, as the construction of vector bundles on the projective space is a much desired objective. We don't know any reference for it.

□

THEOREM 4.5 *Let $\mathbb{P}^2 \cong X \subset \mathbb{P}^5$ a plane in \mathbb{P}^5 . There is no indecomposable rank three vector bundle F on \mathbb{P}^5 with a section having as vanishing scheme a triple structure on X .*

Proof. According to [M2] a triple structure, which is locally a complete intersection, supported by a smooth surface, is similar to a *triple primitive structure* on a curve (in the Bănică-Forster terminology, cf. [BF1], [BF2]). That means that a triple structure Z on X contains a double one Y and one has the defining exact sequences:

$$\begin{aligned} 0 \rightarrow I_Y/I_X^2 \rightarrow I_X/I_X^2 \rightarrow L (= \mathcal{O}_X(r)) \rightarrow 0, \\ 0 \rightarrow I_Z/I_X I_Y \rightarrow I_Y/I_X I_Y \xrightarrow{p} \mathcal{O}_X(2r) \rightarrow 0, \end{aligned} \tag{19}$$

where p is a retract of the natural inclusion $L^2 = I_X^2/I_X I_Y \hookrightarrow I_Y/I_X I_Y$.

CLAIM 4.6 *The above retract does exist.*

Proof of the claim. Denote by G the rank 2 vector bundle I_Y/I_X^2 on X . The canonical exact sequence:

$$0 \rightarrow I_X^2/I_X I_Y \rightarrow I_Y/I_X I_Y \rightarrow I_Y/I_X^2 \rightarrow 0$$

determines an element $\xi \in \text{Ext}^1(G, L^2)$. But $\text{Ext}^1(G, L^2) \cong H^1(G^\vee(2r)) = 0$, as one sees from the exact sequence (19), and so the above exact sequence splits. □

We come back to the proof of our theorem. The proof will be analogous to the proof of the previous result.

The Hilbert polynomial of Z is:

$$\chi_Z(t) = -\frac{3}{2}t^2 + \frac{6r+9}{2}t + \frac{5r^2+9r+6}{2}$$

On the other hand the existence of an exact sequence:

$$0 \rightarrow \Lambda^3 E \rightarrow \Lambda^2 E \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^5} \rightarrow \mathcal{O}_Z \rightarrow 0$$

imply again:

$$\chi_Z(t) = -\frac{c_3}{2}t^2 - \frac{(c_1+6)c_3}{2}t + \frac{(c_2-2c_1^2-18c_1-51)c_3}{2},$$

where c_i are the Chern classes of E . Comparing the two results, one gets:

$$\begin{aligned} c_1 &= 2r - 3 , \\ c_2 &= \frac{19r^2 + 27r + 39}{3} , \\ c_3 &= -3 . \end{aligned}$$

The expression of c_2 shows that r is a multiple of 3. Denote $r = 3R$. Then:

$$\begin{aligned} c_1 &= 6R - 3 , \\ c_2 &= 57R^2 + 27R + 13 , \\ c_3 &= -3 . \end{aligned}$$

This shape of the Chern classes is impossible, because the Hilbert polynomial of E computed by the Riemann-Roch-Hirzebruch Theorem does not take integral values. Namely, its coefficient of $\binom{t+1}{1}$ in the developing in the basis $\binom{t+i}{i}$, $i = 1, \dots, 5$, which is

$$\frac{207R^4 - 1512R^3 - 1845R^2 - 828R - 134}{12}$$

is not in \mathbb{Z} . □

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